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CARDINAL FUNCTIONS ON ORDERED SPACES

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Contents.

Introduction.

In this report we will give an (affirmative) answer to the following questions, raised by M.A. Maurice:

1. Is it true that the Sorgenfrey line S (the reals with the intervals [a,b) for base) is not orderable?

2. Is the product of two separable ordered spaces Lindelöf?

3. Does there exist a Lindelöf ordered space X, such that X × X is not Lindelöf (or even: such that X × X is not normal)?

We prove 1 by means of 2, which comes down to showing:

Each separable ordered space is a paracompact Arhangelskii p-space.

Another proof of 1 can be found in [4], where the following is proved

An ordered space X is metrizable iff the diagonal of X × X is a G_δ.

The reader should use the first section for reference only. In the second section,p.7 and 8,the two maintheorems are formulated. The answer to questions 2 and 3 can be found in the third section, p.9. Finally the fourth section enumerates some properties of the Sorgenfreyline S. Here special attention is paid to the cocompactness of S. Another property of S can be found in [5].

1. Cardinal functions on ordered spaces

In [2] I. Juhasz defines a.o. the following functions which assign to each topological space a certain cardinal number. Let X be a topological space.

1.1 Lindelöfdegree $\mathcal{L}(X) = \min\{\underline{m} \mid$ each open cover of X has a subcover of power $\leq \underline{m}\}$

height = heriditary Lindelöfdegree $h(X) = \min\{\underline{m} \mid \forall S \subset X \;\; \mathcal{L}(S) \leq \underline{m}\}$

spread $s(X) = \sup\{\underline{m} \mid \exists D \subset X \; |D| = \underline{m}$ and D is discrete$\}$

cellularity number = suslinnumber = $c(X) = \sup\{\underline{m} \mid \exists \mathfrak{A} \; |\mathfrak{A}| = \underline{m}$ and $\mathfrak{A}$ is a disjoint family of open subsets of X$\}$

density $d(X) = \min\{\underline{m} \mid \exists D \subset X \;\; |D| = \underline{m}$ and $D^{-} = X\}$

width = heriditary density $z(X) = \min\{\underline{m} \mid \forall S \subset X \;\; d(S) \leq \underline{m}\}$

π-weight $\pi(X) = \min\{\underline{m} \mid \exists \mathcal{L} \;\; |\mathcal{L}| = \underline{m}$ and $\mathcal{L}$ is a family of non-empty open subsets, such that each non-empty open 0 contains a $B \in \mathcal{L}\}$

weight $w(X) = \min\{\underline{m} \mid \exists \mathcal{L} \;\; |\mathcal{L}| = \underline{m}$ and $\mathcal{L}$ is a basis$\}$

character $\chi(X) = \sup_{p \in X} \min\{\underline{m} \mid \exists \mathcal{L} \;\; |\mathcal{L}| = m$ and $\mathcal{L}$ is a neighborhoodbase for $p\}$

pseudocharacter $\psi(X) = \sup_{p \in X} \min\{\underline{m} \mid \exists \mathcal{L} \;\; |\mathcal{L}| = m$ and $\mathcal{L}$ is a family of open sets, such that $\bigcap \mathcal{L} = \{p\}\}$.

Moreover we define for ordered spaces

the number of isolated points

$$i(X) = |\{p \mid p \in X \text{ and } p \text{ is isolated}\}|$$

the number of jumps

$$j(X) = |\{p \mid p \text{ has an immediate successor}\}$$

1.2 In [2] 2.8 it is proved that for any (infinite) linearly ordered topological space

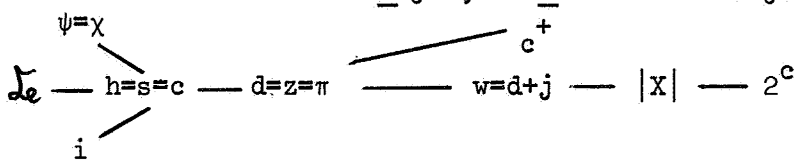
$$\mathcal{L} \leq h=s=c \leq d=z=\pi \leq c^{+} \qquad \text{and}$$

$$d=z=\pi \leq w \leq |X| \leq 2^{c} \; .$$

Also $\psi=\chi \leq c=h=s$.

Moreover it is easily seen that

$i \leq j$, $i \leq h$ and $w = d.j$.



1.3 Let us compute these cardinal functions for some of the well known examples of ordered spaces. We write $\underline{a} = \aleph_0$, and $c = |\mathbb{R}|$.

(a) $[0,m]$ = the set of ordinals less than or equal to the initial ordinal (of power) $\underline{m}$

(b) $[0,\underline{m}^+)$ = the set of ordinals of power $\leq \underline{m}$ (i.e $< \underline{m}^+$)

(c) The real line $\mathbb{R}$.

(d) The long line $[0,\aleph_1) \times [0,1)$ (ordinals x real numbers!) with the lexicographic order.

(e) The Urysohndouble of the unit interval: $[0,1] \times \{0,1\}$ with the lexicographic order.

(f) $[0,1] \times [0,1]$ with lexicographic order.

(g) $[0,1] \times [0,1)$ with lexicographic order.

(h) $[0,1] \times (0,1)$ with lexicographic order is homeomorphic to a topological sum of $\underline{c}$ many copies of $\mathbb{R}$.

(i) A Suslincontinuum.

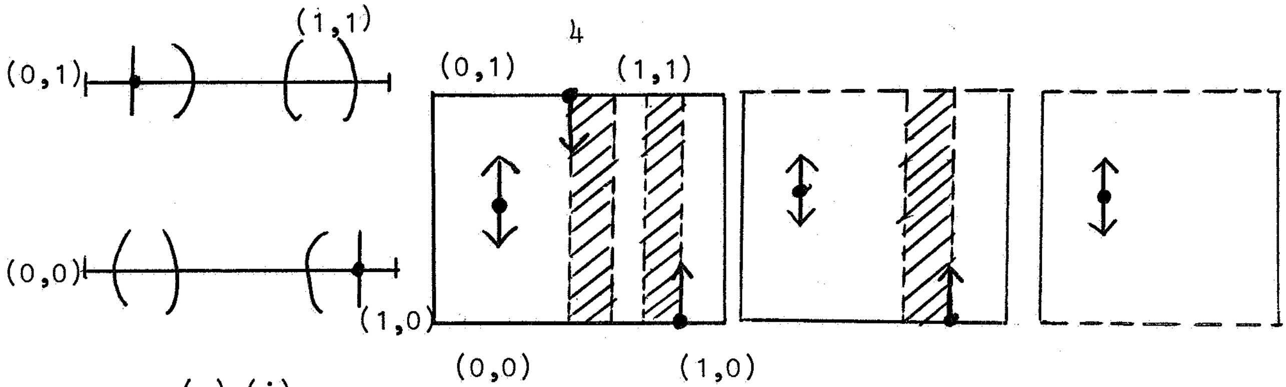
(j) $X \times \{0,1\}$ for an arbitrary ordered space X.

(k) X x Y for ordered spaces X and Y, such that Y has a first but no last element.

(l) X x Y for ordered spaces X and Y, such that Y has both a first and a last element and $|Y| \geq 3$.

(m) X x Y for ordered spaces X and Y such that Y has a first nor last element.

In (j), (k), (l), (m) we take the lexicographic order again. Note that the space of (m) is homeomorphic to a topological sum of $|X|$ copies of Y.



(e),(j) (f),(l) (g),(k) (h),(m)

For each dot a basic neighborhood is indicated.

1.4

space	is the space connected, loc conn. zerodim?	comp	j	i	χ	$\mathfrak{z}$	h=s	d	w	$\|X\|$
(a) $[0,\underline{m}]$	0-dim	yes	$\underline{m}$	$\underline{m}$	$\underline{m}$	$\aleph_0$	$\underline{m}$	$\underline{m}$	$\underline{m}$	$\underline{m}$
(b) $[0,\underline{m}^+)$	0-dim	--	$\underline{m}^+$	$\underline{m}^+$	$\underline{m}$	$\underline{m}^+$	$\underline{m}^+$	$\underline{m}^+$	$\underline{m}^+$	$\underline{m}^+$
(c) $\mathbb{R}$	conn	--	0	0	$\underline{a}$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\underline{c}$
(d) The long line	conn	--	0	0	$\underline{a}$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\aleph_1$	$\underline{c}$
(e) $[0,1] \times \{0,1\}$	0-dim	yes	2	$\underline{c}$	$\underline{a}$	$\aleph_0$	$\aleph_0$	$\aleph_0$	$\underline{c}$	$\underline{c}$
(f) $[0,1] \times [0,1]$	conn	yes	0	0	$\underline{a}$	$\aleph_0$	$\underline{c}$	$\underline{c}$	$\underline{c}$	$\underline{c}$
(g) $[0,1] \times [0,1)$	-	--	0	0	$\underline{a}$	$\aleph_0$	$\underline{c}$	$\underline{c}$	$\underline{c}$	$\underline{c}$
(h) $[0,1] \times (0,1)$	l.c.	--	0	0	$\underline{a}$	$\underline{c}$	$\underline{c}$	$\underline{c}$	$\underline{c}$	$\underline{c}$
(i) Suslin	conn	yes	0	0	$\underline{a}$	$\aleph_0$	$\aleph_0$	$\aleph_1$	$\aleph_1$	$\underline{c}$

1.5 Before we compute the values for X x Y we have to make the following remark. If X is an ordered space, then let $\overrightarrow{X}$ denote the set X with the topology generated by the half open intervals [a,b). This $\overrightarrow{X}$ is homeomorfic to the subset X x {1} of X x {0,1}. With the help of this fact and the clarifying example of $\overrightarrow{X}$ where X = [0,1] x {0,1}, it is easy to verify the following: (Note that, although 1.2 holds for $\overrightarrow{X}$, this is not self evident).

$i(\overrightarrow{X}) = j(X)$

$\mathfrak{z}(\overrightarrow{X}) \leq h(X)$

$w(\overrightarrow{X}) = |X|$

$\chi(\overrightarrow{X}) \leq \chi(X)$

$\phi(\overrightarrow{X}) = \phi(X) + j(X)$ for $\phi \in \{s,c,h,d,z,\pi\}$.

Moreover, $\overset{\longmapsto}{X}$ is of the second Baire category (resp. a Baire space) iff X x {0,1} is of the second category (resp. a Baire space), if X is. However $\overset{\longmapsto}{X}$ may be not Čech complete (and hence neither compact nor locally compact). (cf. 4.1. for hints to the simple proofs). For each X ind $\overset{\longmapsto}{X}$ = 0.

1.6

	(j)	(k)	(l)	(m)
Z =	X x {0,1}	X x Y	X x Y	X x Y
		Y has a first but no last element	Y has both a first and last element and $\lvert Y\rvert \geq 3$	Y has first but no last element
When is Z				
connected	never	1)	iff both X and Y are	never
loc connected	never	2)	iff Y is conn and X l.c.	iff Y is
0-dimensional	always	iff Y is 0-dim	iff Y is 0-dim	iff Y is
compact	iff X is	never, as Y is not compact	iff both X and Y are	never (if X infinite)
i(modulo a finite number)	j(X)	$\lvert X\rvert$. i(Y)[3]	X . i(Y)[3]	$\lvert X\rvert$ .i(Y)
j(modulo a finite number)	$\lvert X\rvert$	$\lvert X\rvert$. j(Y)	j(X)+$\lvert X\rvert$ .j(Y)	$\lvert X\rvert$ .j(Y)
χ	$\chi(X)$	$\leq \chi(X)+\chi(Y)$[4]	$\chi(X) + \chi(Y)$	$\chi(Y)$
$\mathcal{L}$	$\mathcal{L}(X)$	$\leq h(X) + \mathcal{L}(Y)$[4]	$\mathcal{L}(X) + \mathcal{L}(Y)$	$\lvert X\rvert + \mathcal{L}(Y)$
h=s	h(X)+j(X)	$\lvert X\rvert$ + h(Y)	$\lvert X\rvert$ + h(Y)	$\lvert X\rvert$ + h(Y)
d	d(X)+j(X)= = w(X)	$\lvert X\rvert$ + d(Y)	$\lvert X\rvert$ + d(Y)	$\lvert X\rvert$ + d(Y)
w	$\lvert X\rvert$	$\lvert X\rvert$ + w(Y)	$\lvert X\rvert$ + w(Y)	$\lvert X\rvert$ + w(Y)
$\lvert Z\rvert$	$\lvert X\rvert$	$\lvert X\rvert + \lvert Y\rvert$	$\lvert X\rvert + \lvert Y\rvert$	$\lvert X\rvert + \lvert Y\rvert$

---

1) iff Y is connected and each element of X has an immediate successor (i.e. $X \approx \overleftarrow{X}$).

2) iff either (Y is locally connected and each element of X has an immediate predecessor and $\exists y \in Y$ $\{y' \in Y \mid y < y'\}$ is connected) or (Y is connected and $\forall x \in X$ $\exists x' \in X$ $x' < x$ and $\forall x'' \in (x',x)$ x'' has an immediate successor).

3) Unless the initial point of Y is isolated, and is the only isolated point. Then i(XxY) = j(X).

4) Cf [0,m] x (0,1] $\approx$ a topological sum of m copies of ℝ(!) and one copy of (0,1]

2. Perfect mappings and densely ordered spaces. The operators Γ,Δ,E,Z.

2.1 Let $(X,<)$ be an infinite ordered space and $I \subset X$ the set of isolated points of X. We denote the space obtained from X by replacing each isolated point by a unit closed interval $[0,1]$ by $\Gamma(X)$.
Formally: $\Gamma(X) = (X \setminus I) \cup I \times [0,1]$
with the linear order $<^*$ defined by

for $x,y \in X \setminus I$ $\quad x <^* y \quad$ iff $x < y$
for $x,y \in I$, $t,t' \in [0,1]$ $\quad (x,t) <^* (y,t) \quad$ iff $x < y$ or ($x=y$ and $t < t'$).
for $x \in X \setminus I$, $y \in I$, $t \in [0,1]$ $\quad x <^* (y,t)$ iff $x < y$

2.2 Note that $\Gamma(X)$ has no isolated points anymore. All other cardinal-functions defined in §1 however, assume the same values on X and $\Gamma(X)$ with the possible exception of the local weight ψ:
$\psi\,\Gamma(X) = \psi(X) + \aleph_0$, and so $\psi\,\Gamma(X) = \aleph_0 \neq \psi\,X$ iff X is discrete.
Moreover the natural function

$$\pi_\Gamma : \Gamma(X) \to X$$

is perfect.

2.3 Now let $(X,<)$ be a linearly ordered space without isolated points.
We define a relation $\sim$ on X by

$$x \sim y \quad \text{iff there is no } z \in X \text{ in between } x \text{ and } y.$$

Because X has no isolated points, this relation is an equivalence relation, and the equivalence classes consists of one or two points.
Let $<^*$ be the natural order on $X/\sim$, and equip $X/\sim$ with the corresponding order topology. Now we have:

2.4 The identificationmap $\pi_\Delta : X \to X/\sim$ is perfect.
Notation. Put $\Delta(X) = X/_\sim$.

2.5 For a densely ordered space X $j(X) = 0$ and hence $w(X) = d(X)$.

2.6 The following lemma is well known:

If $\phi_s : X_s \to Y_s$, $s \in S$ is a collection of perfect mappings then also $\prod_{s \in S} \phi_s : \prod_{s \in S} X_s \to \prod_{s \in S} Y_s$ is perfect.

2.7 MAINTHEOREM I

If $\{X_s \mid s \in S\}$ is a family of linearly ordered spaces and $\underline{m}$ is a cardinal such that $|S| \leq \underline{m}$ and $d(X_s) \leq \underline{m}$ for each $s \in S$, then the Tychonoffproduct $\prod_{s \in S} X_s$ is $\underline{m}$-Lindelöf, i.e.

$$\mathcal{L}\left(\prod_{s \in S} X_s\right) \leq \underline{m}.$$

Proof. For each $s \in S$ there exist perfect mappings

(i) $\pi_{\Gamma,s}: \Gamma(X_s) \longrightarrow \Delta(\Gamma(X_s))$

and

(ii) $\pi_{\Delta,s}: \Gamma(X_s) \longrightarrow X_s$

So by 2.6 the maps

$$\prod_{s \in S} \pi_{\Gamma,s} : \prod_{s \in S} \Gamma(X_s) \longrightarrow \prod_{s \in S} \Delta(\Gamma(X_s))$$

and

$$\prod_{s \in S} \pi_{\Delta,s} : \prod_{s \in S} \Gamma(X_s) \longrightarrow \prod_{s \in S} X_s$$

are also perfect.

From 2.2 it follows that each $\Gamma(X_s)$ is $\underline{m}$-separable, and since this is a continuous invariant also each X_s is $\underline{m}$-separable. Because $d(\Delta\Gamma X_s) = w(\Delta\Gamma X_s)$ (see 2.5) we find that $\prod_{s \in S} (\Delta\Gamma X_s)$ has a weight $\leq \underline{m}$, and hence is $\underline{m}$-Lindelöf.

For, although $\prod_{s \in S} (\Delta\Gamma X_s)$ cannot be ordered (in general), $\mathcal{L}(Y) \leq w(Y)$ holds for all topological spaces.

Now $\underline{m}$-Lindelöf is an inverse invariant for perfect mappings and an invariant for continuous mappings. Hence also $\prod_{s \in S} \Gamma X_s$ and $\prod_{s \in S} X_s$ are $\underline{m}$-Lindelöf, which completes the proof.

2.8 We define a relation P' on topological spaces by

$XP'Y$ iff there exists a perfect map onto $f: X \to Y$ or $f: Y \to X$.

We make P' into an equivalence relation P, defined on a class $\mathcal{C}$ of topological spaces, called <u>perfect equivalence</u> by

$$XPY \quad \text{iff there exist finitely many } X_1,\ldots,X_n \in \mathcal{C} \text{ such that } X\,P'\,X_1 \text{ and } X_1\,P'\,X_2 \ldots \text{ and } X_n\,P'\,X.$$

In 2.4 we proved the following interesting

MAINTHEOREM II.

<u>In the class of linearly ordered spaces each space X is perfectly equivalent to a space X^* satisfying</u>

<u>(i) X^* has a dense ordertype</u>

<u>(ii) $\phi X = \phi X^*$ for $\phi \in \{ \mathcal{L}, h, c, s, d, z, \pi, \}$ and $wX^* = dX^* = dX$.</u>

2.9 Theorem 2.7 can also be obtained by exploitation of the following constructions. If X is linearly ordered then E(X) is obtained from X by placing a copy of (0,1) in each jump. And Z(X) is obtained from X by only placing a copy of (0,1) in each jump, which has an isolated point for first or last point. Clearly

(i) E(X) is connected, $\mathcal{L}E(X) = \mathcal{L}(X)$, $c\,E(X) = c(X)$, $d\,E(X) = d(X) + j(X) = w(X) = w\,E(X)$.

E(X) contains X as a closed subset. (Example E(Urysohn interval) = $[0,1] \times [0,1]$).

(ii) Z(X) has no isolated points, and contains X as a closed subset. $\mathcal{L}(Z(X)) = \mathcal{L}(X)$, $c(Z(X)) = c(X)$, $d(Z(X)) = d(X)$, $w(Z(X)) = w(X)$.

If we apply the operator Δ to Z(X) we obtain again a perfect image of Z(X) with dense ordertype.

3. Products of separable or Lindelöf ordered spaces.

As a direct consequence of 2.7 we obtain the following

3.1 THEOREM.

The Tychonoffproduct of countably many separable ordered spaces is Lindelöf (and separable)

This theorem does not remain valid if we weaken separable ordered to Lindelöf and ordered, as is shown by the following example:

3.2 Let $X = [0,1] \times (0,1]$ with the lexicographic order. Then the set $\{(x,1) \mid x \in [0,1]\}$ is a closed subset and is homeomorphic to the Sorgenfreyline S. Hence X x X contains a closed copy of S x S. As S x S is not normal this implies that X x X is not normal. Moreover S x S contains a closed discrete subspace of power c, which consequently is also closed and discrete in X x X. Hence X x X is not Lindelöf too.

We indicate two closed discrete subspaces of power c:

$$D_1 = \{((t,1),\ (1-t,1)) \mid t \in [0,1]\}$$
$$D_2 = \{((t,\tfrac{1}{2}),\ (1-t,\tfrac{1}{2})) \mid t \in [0,1]\} \quad .$$

4. The Sorgenfreyline S.

4.1 Let $S = \mathbb{R}$ with $\{[a,b) \mid a,b \in \mathbb{R}\}$ for open base. This space can be considered as the following subspace of the Urysohn double $U = [0,1] \times \{0,1\}$:

(i) $$S \approx (0,1) \times \{1\} \approx (0,1) \times \{0\} \quad .$$

Using (i) it is easily checked that S is
zerodimensional
(hence) completely regular
heriditarily Lindelöf
(hence) heriditarily paracompact
(and) heriditarily normal
heriditarily separable

whilst $\psi(S) = \underline{a}$ and $w(S) = \underline{c}$.
Moreover S is a Baire space, but is not Čech complete.

Proof. The Urysohn space $U = (0,1) \times \{1\} \cup (0,1) \times \{0\} \cup \{(0,0),(0,1),(1,0)$ and $(1,1)\}$ is compact Hausdorff, and hence of the second category. Its only isolated points are $(0,0)$ and $(1,1)$. So if S was of the first category then by (i) so was U. Now each non empty open subset of S contains an open set which is homeomorfic to S and hence is also of the second category. I.e, S is a Baire space. Moreover both $(0,1) \times \{1\}$ and $(0,1) \times \{0\}$ are dense in $U \setminus \{(0,0), (1,1)\}$. So if S was Čech complete then U contained two disjoint dense G_δ's contradicting Baire's category theorem.

4.2 It is well known that $\{(x,-x) \mid x \in \mathbb{R}\}$ is a discrete closed subset of S x S. With this in mind it is easy to deduce the following properties of S x S:

zerodimensional
completely regular
not normal, not Lindelöf, not paracompact
separable, but not heriditary separable

$\psi(S \times S) = a$, $w(S \times S) = c$.

4.3 Because the product of two separable ordered spaces is Lindelöf (2.6 or 3.1) this yields that S cannot be ordered.

4.4 A regular space X is called <u>cocompact</u> (in de Groot's sense) if it has a family $\mathcal{S}$ of closed subsets such that

(i) $\{\text{Int } B \mid B \in \mathcal{S}\}$ constitutes an open (sub)base for X

(ii) Each subfamily of $\mathcal{S}$ with the finite intersection property has a non-empty intersection.

It is easily seen that for the Sorgenfreyline $\mathcal{S} = \{[a,b] \mid a,b \in R\}$ fulfils these conditions. So S is cocompact, which implies too that S is a Baire space (cf. [1]).

A regular space X is called <u>cocompact in van der Slot's sense</u> if there exists a family of open sets $\mathcal{S}$ satisfying

(i)' $\mathcal{S}$ is an open basis

(ii)' For each subfamily of $\mathcal{S}$ with the finite intersection property the closures have a non empty intersection.

As van der Slot has shown, this property is invariant for perfect irreducible maps, which is conjectured to be false for cocompactness. The Sorgenfreyline is the first example of a space which is cocompact, but not cocompact-in-van-der-Slot's-sense. We will prove this now.

Suppose $\mathcal{S}$ is any open basis of S. Let $E = \{\sup B \mid B \in \mathcal{S} \text{ and } \sup B < \infty\}$. Because $\forall x \in \mathbb{R} \ \forall \varepsilon \in \mathbb{R}^+ \ \exists B \in \mathcal{S}$

$$x \in B \subset [x, x+\varepsilon)$$

we find that $\sup B \in (x, x+\varepsilon]$, i.e. E is dense in $\mathbb{R}$ ($\mathbb{R}$ denotes the real line, with Euclidean topology).

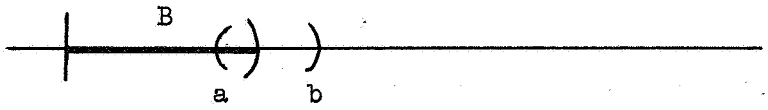
Now put

$$A_n = \{x \in \mathbb{R} \mid \exists B \in \mathcal{S} \ [x,x+\tfrac{1}{n}) \subset B \subset [x,\infty)\} .$$

It is easily checked that

$$\bigcup_{n \in N} A_n = \mathbb{R} .$$

Hence by Baire's category theorem for $\mathbb{R}$ $\exists n$, $\exists a, b \in \mathbb{R}$ $a < b$ and A_n is dense in (a,b).



Moreover, as E is dense $\exists B \in \mathcal{L}$ $b = \sup E \in (a,b)$. Now choose $x_k \in A_n \cap (a,b)$ such that $x_k \uparrow b$ for $k \to \infty$, and $s - x_k < \frac{1}{n}$ for all k. For each k we can find a $B_k \in \mathcal{L}$ such that $b \in [x_k, x_k + \frac{1}{n}) \subset B_k \subset [x_k, \infty)$. Now it is easily seen that

$$\{B, B_1, B_2, B_3, \ldots\}$$

is a centered family, and $\bigcap_{n \in N} B_n^- \subset [B, \infty)$, but $b \notin B^-$.

Hence

$$B^- \cap B_1^- \cap B_2^- \cap \ldots = \phi.$$

Thus $\mathcal{L}$ cannot satisfy (ii)!

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